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Tangent Bundle of globally symplectomorphic inhomogeneous Einstein manifold to R^{2n}

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Abstract:

We find out properties of tangent bundle of explicit global symplectic coordinates for the Calabi's inhomogeneous Kähler–Einstein metric on tubular domains.

Keywords: Tangent Bundle, Kähler metrics, Calabi's inhomogeneous Einstein metric

1 Introduction:

Let ω be a Kähler form on an n-dimensional complex manifold M diffeomorphic to $R^{2n} \cong C_n$. One basic and fundamental question from the symplectic point of view is to understand when (M, ω) admits global symplectic coordinates, i.e. when there exists a global diffeomorphism Ψ: M $\rightarrow R^{2n}$ such that $\Psi^*\omega_0 = \omega$, where $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dx_j$ is the standard symplectic form on R^{2n}. Therefore it is natural to look for sufficient conditions, related to the Riemannian or to the complex structure of the manifold involved, which assure the existence of global symplectic coordinates.

D. McDuff [7] proved a global version of Darboux theorem for complete and simply-connected Kähler manifolds with non-positive sectional curvature and in [4] the first author and A. Di Scala provide a construction of a global symplectomorphism from bounded symmetric domains equipped with the Bergman metric to (R^{2n}, ω_0), using the theory of Jordan triple systems. Constructions of explicit global symplectic coordinates on some complex domains explicit global symplectic coordinates for the Calabi's inhomogeneous Kähler–Einstein form ω on the complex tubular domains

$M = \frac{1}{2} D_C \oplus iR^n \subset C^n$, $n \geq 2$, where $D_a \subset R^n$ is the open ball of R^n centered at the origin and of radius a.

In this paper we study tangent bundle of globally symplectomorphic inhomogeneous Einstein manifold to R^{2n}

Theorem 1. For all $n \geq 2$, the Kähler manifold (M, ω) is globally symplectomorphic to (R^{2n}, ω_0) via the map:

$$\Phi: M \rightarrow R^n \oplus iR^n \cong R^{2n}, (x, y) \rightarrow (\text{grad } f, y), \quad (1)$$

Where f: $D_a \rightarrow R$, $x = (x_1, x_n) \rightarrow f(x)$ is a Kähler potential for ω,

I.e. $\omega = \partial \bar{\partial} f$, and grad $f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots\ldots \frac{\partial f}{\partial x_n},)$

Notice that in [2, p. 23] Calabi provides an explicit formula for the curvature tensor of (M, g) (he needs this formula to show that the metric g associated to the Kähler form ω is not locally homogeneous), consequently, it is not clear if g satisfies or not the assumptions of McDuff's theorem, namely if its sectional curvature is non positive. Nevertheless, it is worth pointing out that the proof of McDuff's result is telling us that there exist global symplectic coordinates, but it is not giving any criterium to compute explicitly them as we did in Theorem 1.

Result should be used to give an explicit description of all Langragian submanifolds of (M, ω) which have classically played an important role in symplectic geometry.

2. Calabi's metric

Consider the complex tubular domain $M = \frac{1}{2} D_a \oplus iR^n \subset C^n$, as in the introduction. Let g be the metric on $M \subset C^n$ whose
associated Kähler form is given by:

$$\omega = \frac{i}{2}\partial\bar{\partial} f(z_1 + \bar{z}_1, \ldots\ldots, z_n + \bar{z}_n) \qquad (2)$$

where $f : D_a \rightarrow R$ is a radial function $f(x_1, \ldots, x_n) = Y(r)$,
for $r = (\sum_{j=1}^{n} x_j^2)^{1/2}$, for $x_j = (z_j + \bar{z}_j)/2$, $y_j = (z_j - \bar{z}_j)/2$,
that satisfies the differential equation:

$$(Y'/r)^{n-1} Y'' = e^Y , \qquad (3)$$

with initial conditions: $Y'(0) = 0$, $Y''(0) = e^{Y(0)/n}$. (4)

In [2], Calabi proved that the Kähler metric g so defined is smooth, Einstein, complete and not locally homogeneous.
This was indeed the first example of such a metric. The reader is also referred to [8] for an alternative and easier proof of the fact that this metric is complete but not locally homogeneous.

3. Tangent Bundle

Vector field on an abstract manifold M as an assignment of a tangent vector Y (p) to each point p ∈ M. Thus Y is a map from M into the set of all tangent vectors at all points of M. It turns out that this set of tangent vectors can be given a differential structure of its own, by means of which the smoothness of Y can be elegantly expressed.

Definition 3.1

The tangent bundle of M is the union $TM = \cup_{p\in M} T_pM$ of all tangent vectors at all points.
For a given element X ∈TM the point p ∈ M for which $X \in T_pM$ is called the base point. The map π: TM → M which assigns p to X, is called the projection.
Let $\Phi: M \rightarrow R^n \oplus iR^n$ be a smooth map between manifold, then its differential dΦ maps T_pM into $T_{\Phi(p)}R$ For each p∈M. The collection of these maps for all p∈M, is a map from TM to TR which is denoted by gradΦ.
In the special case where $\Phi: M \rightarrow R^n \oplus iR^n$ maps an open set $M \subset R^n$ smoothly into R^{2n}, the differential $d\Phi: M \rightarrow R^n \oplus iR^n$ is the map given by dΦ(x,y) = (Φ(x), gradf(x)y). Observe that this is a smooth map.

Theorem 3.1

Let M be an abstract manifold. The collection consisting of the maps $d\Phi: M \rightarrow R^n \oplus iR^n$, for all charts Φ in an atlas on M, is an atlas for a structure of an abstract manifold on the tangent bundle TM. With this structure the projection π: TM → M is smooth, and a vector field Y on M is smooth if and only if it is smooth as a map from M to TM.

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