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Certain Class of Analytic Functions With Respect to Symmetric Points

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Abstract. In this paper, we obtain sharp results for coefficient inequality, closure theorem and other related results with respect to symmetric for the classes $S_T^*(\alpha,\beta,\xi,\gamma)$ and $C_T(\alpha,\beta,\xi,\gamma)$ and shall be denoted by $S_{ST}^*(\alpha,\beta,\xi,\gamma)$ and $C_{ST}^*(\alpha,\beta,\xi,\gamma)$.

Key Words and Phrases. Analytic functions; Negative coefficient; Starlike functions; Convex functions; Symmetric Points.

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1 Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk

$$U = \left\{ z : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1 \right\}.$$

We denote by $S^*(\alpha)$ and $C(\alpha)$ the subclasses of $\mathcal{A}$ consisting of all function which are starlike and convex of order α $(0 \leq \alpha < 1)$, respectively, in U, that is

$$S^*(\alpha) = \left\{ f \in S; \mathrm{Re}\left(z\frac{f'(z)}{f(z)} \right) > \alpha; 0 \leq \alpha < 1, z \in U \right\}$$

and

$$C(\alpha) = \left\{ f \in S; \mathrm{Re}\left(1 + z\frac{f''(z)}{f'(z)} \right) > \alpha; 0 \leq \alpha < 1, z \in U \right\}.$$

We say that the function $f(z)$ is in the class $S(\alpha,\beta,\xi,\gamma)$ if and only if

$$\left| \frac{z\frac{f'(z)}{f(z)} - 1}{2\xi\left(z\frac{f'(z)}{f(z)} - \alpha \right) - \gamma\left(z\frac{f'(z)}{f(z)} - 1 \right)} \right| < \beta$$

for $|z| < 1$ where $0 < \beta \leq 1;\ \frac{1}{2} \leq \xi \leq 1;\ 0 \leq \alpha \leq \frac{1}{2};\ \frac{1}{2} < \gamma \leq 1.$

A function f is said to belong to the class $C(\alpha,\beta,\xi,\gamma)$ if and only if $zf' \in S^*(\alpha,\beta,\xi,\gamma)$.

Many researchers have introduced and investigated several subclasses of analytic function class $\mathcal{A}$ (see, for example [6], [7] and [11]). Various subclasses of $\mathcal{A}$ were introduced and some geometric properties of these subclasses were investigated in several studies (see [3], [5] and [12]).

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad (a_n \geq 0). \tag{1.2}$$

We denote by $S_T^*(\alpha,\beta,\xi,\gamma)$ and $C_T(\alpha,\beta,\xi,\gamma)$ the classes obtained by taking intersection, respectively, of the classes $S(\alpha,\beta,\xi,\gamma)$ and $C(\alpha,\beta,\xi,\gamma)$, that is

$$S_T^*(\alpha,\beta,\xi,\gamma) = S^*(\alpha,\beta,\xi,\gamma) \cap T \text{ and } C_T(\alpha,\beta,\xi,\gamma) = C(\alpha,\beta,\xi,\gamma) \cap T.$$

These functions are called starlike with respect to symmetric points and were introduced by Shaqsi and Darus [1], [2], Ghanim and Darus [8], Sakaguchi [9] and Sudharsan *et al.* [13]. Recently, El-Ashwah and Thomas [4] have introduced two function classes, namely the class of functions starlike with respect to conjugate points and the class of functions starlike with respect to symmetric conjugate points.

In this paper, we obtain sharp results for coefficient inequality, closure theorem and other related results with respect to symmetric for the classes $S_T^*(\alpha,\beta,\xi,\gamma)$ and $C_T(\alpha,\beta,\xi,\gamma)$ and shall be denoted by $S_{ST}^*(\alpha,\beta,\xi,\gamma)$ and $C_{ST}^*(\alpha,\beta,\xi,\gamma)$.

We say that the function f is in the class $S_{ST}^*(\alpha,\beta,\xi,\gamma)$ if and only if

$$\left| \frac{z\frac{f'(z)}{f(z)-f(-z)} - 1}{2\xi\left(z\frac{f'(z)}{f(z)-f(-z)} - \alpha\right) - \gamma\left(z\frac{f'(z)}{f(z)-f(-z)} - 1\right)} \right| < \beta. \tag{1.3}$$

Next, we find the coefficient inequality for the class $S_{ST}^*(\alpha,\beta,\xi,\gamma)$.

2 Coefficient inequalities

Theorem 2.1 *function $f \in T$ given by (1.2) is in the class $S_{ST}^*(\alpha,\beta,\xi,\gamma)$ if and only if;*

$$\sum_{n=2}^{\infty} \left[(n-2) - \beta(\gamma n - 2\gamma + 4\xi\alpha - 2n\xi)\right][a_n] \leq 4\beta\xi(1-\alpha).$$

Proof. Suppose,

$$\sum_{n=2}^{\infty} \left[(n-2) - \beta(\gamma n - 2\gamma + 4\xi\alpha - 2n\xi)\right][a_n] \leq 4\beta\xi(1-\alpha).$$

From (1.3) we have

$$|zf'(z) - (f(z) - f(-z))| -$$

$$\beta\left|2\xi\left(zf'(z) - \alpha\left(f(z) - f(-z)\right)\right) - \gamma\left(zf'(z) - \left(f(z) - f(-z)\right)\right)\right| < 0.$$

Given that

$$\left|\sum_{n=2}^{\infty} (n-2)\,|a_n|\right| -$$

$$\beta\left|4\xi(1-\alpha) + \sum_{n=2}^{\infty} (\gamma n - 2\gamma + 4\xi\alpha - 2n\xi)\right| < 0$$

for $|z| = r < 1$; then the condition (2.1) is bounded above by

$$\sum_{n=2}^{\infty} (n-2)\,|a_n|\,r^n - 4\beta\xi(1-\alpha) - \beta \sum_{n=2}^{\infty} (\gamma n - 2\gamma + 4\xi\alpha - 2n\xi)\,|a_n|\,r^n$$

$$= \sum_{n=2}^{\infty} \{(n-2) - \beta(\gamma n - 2\gamma + 4\xi\alpha - 2n\xi)\}\,|a_n|\,r^n - 4\beta\xi(1-\alpha)$$

$$\le \sum_{n=2}^{\infty} \{(n-2) - \beta(\gamma n - 2\gamma + 4\xi\alpha - 2n\xi)\}\,|a_n| - 4\beta\xi(1-\alpha) \le 0$$

Therefore $f(z) \in S^*_{ST}(\alpha, \beta, \xi, \gamma)$.

Now we prove the converse.
Let

$$\left| \frac{z\frac{f'(z)}{f(z)-f(-z)} - 1}{2\xi\left(z\frac{f'(z)}{(f(z)-f(-z))} - \alpha\right) - \gamma\left(z\frac{f'(z)}{f(z)-f(-z)} - 1\right)} \right|$$

$$\left| \frac{\sum\limits_{n=2}^{\infty} (n-2)a_n z^n}{4\xi(1-\alpha) + \sum\limits_{n=2}^{\infty} (\gamma n - 2\gamma + 4\xi\alpha - 2n\xi)a_n z^n} \right| < \beta$$

as $|Re(z)| \le |z|$ for all z, we have

$$\text{Re}\left\{ \frac{\sum\limits_{n=2}^{\infty} (n-2)\,a_n z^n}{4\xi(1-\alpha) + \sum\limits_{n=2}^{\infty} (\gamma n - 2\gamma + 4\xi\alpha - 2n\xi)a_n z^n} \right\}.$$

We choose the values of z on real axis such that $\frac{zf'(z)}{f(z)-f(-z)}$ is real and upon clearing the denominator of above expression and letting $z \longrightarrow 1$ through real values we obtain

$$\sum_{n=2}^{\infty} \{(n-2) - \beta(\gamma n - 2\gamma + 4\xi\alpha - 2n\xi)\}\,|a_n| \le 4\beta\xi(1-\alpha).$$

■

Corollary 2.2 *If $f \in S^*_{ST}(\alpha, \beta, \xi, \gamma)$, then*

$$|a_n| \le \frac{4\beta\xi(1-\alpha)}{\{(n-2) - \beta(\gamma n - 2\gamma + 4\xi\alpha - 2n\xi)\}} \qquad for \quad n = 2, 3, \ldots.$$

Equality holds for

$$f(z) = z - \frac{4\beta\xi(1-\alpha)}{\{(n-2) - \beta(\gamma n - 2\gamma + 4\xi\alpha - 2n\xi)\}} z^n.$$

Corollary 2.3 *If $f(z) \in S^*_{ST}(\alpha, \beta, \xi, 1)$, we get*

$$|a_n| \le \frac{4\beta\xi(1-\alpha)}{\{(n-2) - \beta(n - 2 + 4\xi\alpha - 2n\xi)\}} \qquad for \quad n = 2, 3, \ldots.$$

Equality holds for;

$$f(z) = z - \frac{4\beta\xi(1-\alpha)}{\{(n-2) - \beta(n - 2 + 4\xi\alpha - 2n\xi)\}} z^n.$$

Corollary 2.4 *If* $f(z) \in S^*_{ST}(\alpha, \beta, 1, 1)$*, we get*

$$f(z) = z - \frac{4\beta(1-\alpha)}{\{(n-2) - \beta(4\alpha - n - 2)\}} z^n.$$

Equality holds for;

$$f(z) = z - \frac{4\beta(1-\alpha)}{\{(n-2) - \beta(4\alpha - n - 2)\}} z^n.$$

Corollary 2.5 $f(z) \in S^*_{ST}(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty} |(n-2\alpha)\, a_n| z^n \leq (1-\alpha).$$

Theorem 2.6 *A function* f *given by (1.2) is in* $C_{ST}(\alpha, \beta, \xi, \gamma)$ *if and only if*

$$\sum_{n=2}^{\infty} n\left[(n-2) - \beta(\gamma n - 2\gamma + 4\xi\alpha - 2n\xi)\right] |a_n| \leq 4\beta\xi(1-\alpha)$$

Proof. The proof of this theorem is analogous to that of Theorem 1 because a function $f(z) \in C_{ST}(\alpha, \beta, \xi, \gamma)$ if and only if $zf' \in S^*_{ST}(\alpha, \beta, \xi, \gamma)$ so it is enough that a_n in the Theorem 1 replace with na_n. ■

Corollary 2.7 *If* $f(z) \in C_{ST}(\alpha, \beta, \xi, \gamma)$*, then*

$$|a_n| \leq \frac{4\beta\xi(1-\alpha)}{n\{(n-2) - \beta(\gamma n - 2\gamma + 4\xi\alpha - 2n\xi)\}} \qquad for \quad n = 2, 3, \ldots.$$

Equality holds for;

$$f(z) = z - \frac{4\beta\xi(1-\alpha)}{n\{(n-2) - \beta(\gamma n - 2\gamma + 4\xi\alpha - 2n\xi)\}} z^n.$$

Corollary 2.8 *If* $f(z) \in C_{ST}(\alpha, \beta, \xi, 1)$*, we get*

$$|a_n| \leq \frac{4\beta\xi(1-\alpha)}{n\{(n-2) - \beta(n - 2 + 4\xi\alpha - 2n\xi)\}} \qquad for \quad n = 2, 3, \ldots \; .$$

Equality holds for

$$f(z) = z - \frac{4\beta\xi(1-\alpha)}{n\{(n-2) - \beta(n - 2 + 4\xi\alpha - 2n\xi)\}} z^n.$$

Corollary 2.9 *If* $f(z) \in C_{ST}(\alpha, \beta, 1, 1)$*, we get*

$$f(z) = z - \frac{4\beta(1-\alpha)}{n\{(n-2) - \beta(4\alpha - n - 2)\}} z^n.$$

Equality holds for

$$f(z) = z - \frac{4\beta(1-\alpha)}{n\{(n-2) - \beta(4\alpha - n - 2)\}} z^n.$$

Corollary 2.10 *If* $f(z) \in C_{ST}(\alpha)$ *that is starlike with respect to symmetric point of order* α *if and only if*

$$\sum_{n=2}^{\infty} |n(n-2\alpha)\, a_n| z^n \leq (1-\alpha).$$

3 Closure Theorem

Theorem 3.1 *Let $f_1(z) = z$ and*

$$f_n(z) = z - \frac{4\beta\xi(1-\alpha)}{[(n-2) - \beta(n - 2\gamma + 4\xi\alpha - 2n\xi)]} z^n \qquad for \qquad n = 2, 3, 4, ... \ .$$

Then $f(z) \in C_{ST}(\alpha, \beta, \xi, \gamma)$ if and only if $f(z)$ can be expressed in the forms,

$$f(z) = f_1(z) - \sum_{n=2}^{\infty} \lambda_n f_n(z) \qquad where \ \lambda_n \geq 0 \qquad and \ \sum \lambda_n = 1.$$

Proof. Suppose,

$$f(z) = z - \sum_{n=2}^{\infty} \lambda_n f_n(z)$$

$$= z - \sum_{n=2}^{\infty} \frac{4\beta\xi(1-\alpha)}{[(n-2) - \beta(n - 2\gamma + 4\xi\alpha - 2n\xi)]} z^n.$$

Then

$$\sum_{n=2}^{\infty} \frac{\lambda_n 4\beta\xi(1-\alpha)}{[(n-2) - \beta(n - 2\gamma + 4\xi\alpha - 2n\xi)]} \times$$

$$\frac{[(n-2) - \beta(n - 2\gamma + 4\xi\alpha - 2n\xi)]}{4\beta\xi(1-\alpha)}$$

$$= \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1.$$

Therefore $f(z) \in C_{ST}(\alpha, \beta, \xi, \gamma)$

Conversely, suppose $f(z) \in C_{ST}(\alpha, \beta, \xi, \gamma)$ then remark of Theorem 2.1 gives us

$$|a_n| \leq \frac{4\beta\xi(1-\alpha)}{\{(n-2) - \beta(\gamma n - 2\gamma + 4\xi\alpha - 2n\xi)\}} \qquad for \quad n = 2, 3, ... \ .$$

$$\lambda_n = \left[\frac{(n-2) - \beta\,(n\gamma - 2\gamma + 4\xi\alpha - 2n\xi)}{4\beta\xi\,(1-\alpha)}\right] |a_n|\,.$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n.$$

Then

$$(z) = -\sum_{n=2}^{\infty} \lambda_n f_n(z).$$

■

Corollary 3.2 *If $f_1(z) = z$ and*

$$f_n(z) = z - \frac{4\beta\xi(1-\alpha)}{[(n-2) - \beta(n-2+4\xi\alpha - 2n\xi)]} z^n \qquad for \qquad n = 2, 3, 4, ...$$

Then $f(z) \in C_{ST}(\alpha, \beta, \xi, 1)$ if and only if $f(z)$ can be expressed in the form

$$f(z) = f_1(z) - \sum_{n=2}^{\infty} \lambda_n f_n(z) \qquad where\ \lambda_n \geq 0 \qquad n = 1, 2, ... \quad .$$

$$\sum_{n=2}^{\infty} \lambda_n = 1.$$

Corollary 3.3 *If $f_1(z) = z$ and*

$$f_n(z) = z - \frac{4\beta(1-\alpha)}{[(n-2) - \beta(n-2+4\alpha - 2n)]} z^n \qquad for \qquad n = 2, 3, 4, ...$$

Then $f(z) \in C_{ST}(\alpha, \beta)$ if and only if $f(z)$ can be expressed in the form

$$f(z) = f_1(z) - \sum_{n=2}^{\infty} \lambda_n f_n(z) \qquad where\ \lambda_n \geq 0 \qquad n = 1, 2, ... \quad .$$

$$\sum_{n=2}^{\infty} \lambda_n = 1.$$

If $f_1(z) = z$ and

$$f(z) = z - \left|\frac{1}{n}\right| z^n.$$

Then $f(z) \in C_{ST}(0, 1, 1, 1)$ if and only if ,$f(z)$ can be expressed in the form

$$f(z) = f_1(z) - \sum_{n=2}^{\infty} \lambda_n f_n(z) \qquad where\ \lambda_n \geq 0 \qquad n = 1, 2, ...$$

$$\sum_{n=2}^{\infty} \lambda_n = 1.$$

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